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ON THE DISTRIBUTION OF THE PRODUCT OF TWO GAMMA VARIATES

Palmer R. Schlegel

September 1977

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USA ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
USA BALLISTIC RESEARCH LABORATORY
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In this paper Mellin tranforms are used to determine the distribution of the product of two random variables, each of which is distributed by gamma functions and parameterized by an integer n. Furthermore, a recursive relationship is developed with respect to the parameter n to facilitate the evaluation of the resulting distribution.

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I. INTRODUCTION

If one is interested in the distribution of the sum of two random variables, an immediate approach would be to use Fourier transforms. It is well known that the Fourier transform of the density function of the sum of two random variables is equal to the product of the Fourier transforms of the density functions of the respective random variables.

To determine the distribution of the product of two random variables, an analogous approach, namely, the application of Mellin transforms, will be given in this paper. In addition, a recursive relationship will be developed to evaluate the resulting distribution.

II. DEVELOPMENT OF DISTRIBUTION

If the density function of a random variable vanishes in the interval $(-\infty,0)$, then the Mellin transform, which is defined by

$$M(f)(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{1}$$

exists. Furthermore, it is known and easy to show that the Mellin transform of the density function of the product of two random variables is the product of the Mellin transforms of each denity function.

Let X, Y and XY be random variables and f_X, f_Y and f_{XY} denote the respective density functions, where

$$f_X(x) = \frac{e^{-x}x^{n-2}}{(n-2)!} \tag{2}$$

and

$$f_Y(y) = \frac{e^{-y}y^{n-1}}{(n-1)!} \quad . \tag{3}$$

Then

$$M(f_X)(s) = \int_0^\infty \frac{e^{-x}x^{n-2}x^{s-1}}{(n-2)!}\,dx = \frac{\Gamma(s+n-2)}{(n-2)!} \tag{4}$$

and

$$M(f_Y)(s) = \int_0^\infty \frac{e^{-y} y^{n-1} y^{s-1}}{(n-1)!} dy = \frac{\Gamma(s+n-1)}{(n-1)!} , \tag{5}$$

where $\Gamma(u)$ is the gamma function (see [1]). Therefore, from the relationship of Mellin transforms, we have

$$M(f_{XY})(s) = \frac{\Gamma(s+n-2)\Gamma(s+n-1)}{(n-2)!(n-1)!} . \tag{6}$$

From the inverse Mellin transform (see [2]),

$$f_{XY}(z) = \frac{2(\sqrt{z})^{2n-3} K_1(2\sqrt{z})}{(n-1)!(n-2)!} , \tag{7}$$

where $K_1(u)$ is a first order modified Bessel function of the third kind.

The distribution function of the random variable $Z = XY$ is given by

$$P(Z < z) = \int_0^z f_{XY}(u) du = \frac{[2]^{3-2n}}{(n-1)!(n-2)!} \int_0^{2\sqrt{z}} u^{2n-2} K_1(u) du. \tag{8}$$

III. EVALUATION OF THE DISTRIBUTION

The distribution (8) is parameterized by an integer n. We will now proceed to develop a recursive relationship in terms of the parameter that will facilitate the evaluation of the distribution.

Define

$$P_n(z) = \frac{[2]^{3-2n}}{(n-1)!(n-2)!} \int_0^{2\sqrt{z}} u^{2n-2} K_1(u) du. \tag{9}$$

To simplify notation, we will consider

$$I_n = \int_0^a u^{2n-2} K_1(u) du. \tag{10}$$

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[1] This problem was suggested to the author by D. Clark, AMSAA.

[2] Erdelyi, A., et al, Editor, "Table of Integral Transforms", Bateman Manuscript Project, Vol. I, McGraw-Hill Book Co., Inc., New York, 1954

From the identity (see [3])

$$K'_m(x) = -K_{m+1}(x) + \frac{m}{x} K_m(x), \tag{11}$$

we have, where from (11) $K_1(u) = -K'_0(u)$, by integration by parts

$$I_n = -\int_0^a u^{2n-2} K'_0(u)du = -a^{2n-2}K_0(a) + (2n-2)\int_0^a u^{2n-3}K_0(u)du. \tag{12}$$

From (11), where m = -1 and the fact that $K_m(x) = K_{-m}(x)$, we have

$$K_0(x) = -K'_1(x) - x^{-1}K_1(x). \tag{13}$$

If we substitute (13) into (12) and integrate by parts, we obtain

$$I_n = -a^{2n-2}K_0(a) - (2n-2)\int_0^a [u^{2n-3}K'_1(u) + u^{2n-4}K_1(u)]du$$

$$= -a^{2n-2}K_0(a) - (2n-2)a^{2n-3}K_1(a) +$$

$$+ (2n-2)(2n-4)\int_0^a u^{2n-4}K_1(u)du. \tag{14}$$

Therefore, from (9)

$$P_n(z) = \frac{[2]^{3-2n}}{(n-1)!(n-2)!}\left[-(2\sqrt{z})^{2n-2}K_0(2\sqrt{z}) - (2n-2)(2\sqrt{z})^{2n-3}K_1(2\sqrt{z}) + \right.$$

$$\left. + (2n-2)(2n-4)\int_0^{2\sqrt{z}} u^{2n-4}K_1(u)du\right]$$

$$= P_{n-1}(z) - \frac{(\sqrt{z})^{2n-3}}{(n-1)!(n-2)!}\left[2\sqrt{z}\ K_0(2\sqrt{z}) + (2n-2)K_1(2\sqrt{z})\right], \tag{15}$$

for n > 2, where

$$P_2 = 1 - 2zK_2(2\sqrt{z}). \tag{16}$$

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[3]Abramowitz, Milton and Stegun, Irene A., Editors, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", National Bureau of Standards, Applied Mathematics Series-55, 1964.

and $K_m(x)$, m = 0, 1, 2, are mth order modified Bessel functions of the third kind.

IV. CONCLUSION

The original problem, as presented to the author, was to construct a double-entry table of the distribution over the variable z and the parameter n. A numerical scheme involved an approximation of a double integration over an infinite limit had been programmed for BRLESC. This procedure was estimated to take an average of five minutes of computer time to evaluate a single point in the double-entry table. Equations (15) and (16) reduced this to a table look-up and a simple calculation with a hand calculator.

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